Laguerre polynomials and the inverse Laplace transform using discrete data

Tran Ngoc Lien[1] , Dang Duc Trong[2] and Alain Pham Ngoc Dinh[3]

Abstract. We consider the problem of finding a function defined on $(0, \infty)$ from a countable set of values of its Laplace transform. The problem is severely ill-posed. We shall use the expansion of the function in a series of Laguerre polynomials to convert the problem in an analytic interpolation problem. Then, using the coefficients of Lagrange polynomials we shall construct a stable approximation solution. Error estimate is given. Numerical results are produced.

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1. Introduction

Let $L_\rho^2(0, \infty)$ be the space of real Lebesgue measurable functions defined on $(0, \infty)$ such that

$$\|f\|_{L_\rho^2}^2 \equiv \int_0^\infty |f(x)|^2 e^{-x} dx < \infty.$$

This is a Hilbert space corresponding to the inner product

$$< f, g >= \int_0^\infty f(x)g(x)e^{-x} dx.$$

We consider the problem of recovering a function $f \in L_\rho^2(0, \infty)$ satisfying the equations

$$\mathcal{L}f(p_j) \equiv \int_0^\infty e^{-p_j x} f(x) dx = \mu_j \qquad (DIL)$$

where $p_j \in (0, \infty), j = 1, 2, 3, ...$

Generally, we have the classical problem of finding a function $f(x)$ from its given image $g(p)$ satisfying

$$\mathcal{L}f(p) \equiv \int_0^\infty e^{-px} f(x) dx = g(p), \qquad (1)$$

where p is in a subset ω of the complex plane. We note that $\mathcal{L}f(p)$ is usually an analytic function on a half plane $\{Re\ p > \alpha\}$ for an appropriate real number α. Frequently, the image of a Laplace

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[1]Faculty of Sciences, Cantho University, Cantho, Vietnam. Email: tnlien@ctu.edu.vn.

[2]Department of Mathematics and Computer Sciences, Hochiminh City National University, 227 Nguyen Van Cu, Hochiminh City, Vietnam. Email: ddtrong@mathdep.hcmuns.edu.vn.

[3]Mapmo UMR 6628, Université d'Orleans, BP 6759, 45067 Orleans Cedex, France. Email: alain.pham@math.cnrs.fr

transform is known only on a subset ω of the right half plane $\{Re\ p > \alpha\}$. Depending on the set ω, we shall have appropriate methods to construct the function f from the values in the set

$$\{\mathcal{L}f(p):\ p \in \omega\}.$$

Hence, there are no universal methods of inversion of the Laplace transform.

If the data $g(p)$ is given as a function on a line $(-i\infty + a, +i\infty + a)$ (i.e., $\omega = \{p:\ p = a + iy,\ y \in \mathbf{R}\}$) on the complex plane then we can use the Bromwich inversion formula ([26], p. 67) to find the function $f(x)$.

If $\omega \subset \{p \in \mathbf{R}:\ p > 0\}$ then we have the problem of real inverse Laplace transform. The right hand side is known only on $(0, \infty)$ or a subset of $(0, \infty)$. In this case, the use of the Bromwich formula is therefore not feasible. The literature on the subject is impressed in both theoretical and computational aspects (see, e.g. [2,3,10,16,18,22]). In fact, if the data $g(p)$ is given exactly then, by the analyticity of g, we have many inversion formulas (see,e.g., [3,7,8,20,21,23]). In [3], the author approximate the function f by

$$f(t) \cong \sum_{k=0}^{N} b_k(a) d^k (e^x g(e^x))/dx^k$$

where $b_k(a)$ are calculated and tabulated regularization coefficients and g is the given Laplace transform of f. Another method is developped by Saitoh and his group ([4,5,20,21]), where the function f is approximated by integrals having the form

$$u_N(t) = \int_0^\infty g(s) e^{-st} P_N(st) ds, \qquad N = 1, 2, ...$$

where P_N is known (see [5]). Using the Saitoh formula, we can get directly error estimates.

However, in the case of unexact data, we have a severely trouble by the ill-posedness of the problem. In fact, a solution corresponding to the unexact data do not exist if the data is nonsmooth, and in the case of existence, these do not depend continuously on the given data (that are represented by the right hand side of the equalities). Hence, a regularization method is in order. In [7], the authors used the Tikhonov method to regularize the problem. In fact, in this method, we can approximate u_0 by functions u_β satisfying

$$\beta u_\beta + \mathcal{L}^* \mathcal{L} u_\beta = \mathcal{L}^* g, \qquad \beta > 0.$$

Since $\mathcal{L}$ is self-adjoint (cf. [7]), the latter equation can be written as

$$\beta u_\beta + \int_0^\infty \frac{u_\beta(s)}{s+t} ds = \int_0^\infty e^{-st} g(s) ds.$$

The latter problem is well-posed.

Although the inverse Laplace transform has a rich literature, the papers devoted to the problem with discrete data are scarce. In fact, from the analyticity of $\mathcal{L}f(p)$,if $\mathcal{L}f(p)$ is known on a countable subset of $\omega \subset \{Re\ p > \alpha\}$ accumulating at a point then $\mathcal{L}f(p)$ is known on the whole $\{Re\ p > \alpha\}$. Hence, generally, a set of discrete data is enough for constructing an approximation function of f. It is a moment problem. In [15], the authors presented some theorems on the stabilization of the inverse Laplace transform. The Laplace image is measured at N points to within some error ϵ. This is achieved by proving parallel stabilization results for a related Hausdorff moment problem. For a construction of an approximate solution of (DIL), we note that the sequence of functions $(e^{-p_j x})$ is

(algebraically) linear independent and moreover the vector space generated by the latter sequence is dense in $L^2(0, \infty)$. The method of truncated expansion as presented in ([6], Section 2.1) is applicable and we refer the reader to this reference for full details. In [11, 13], the authors convert (DIL) into a moment problem of finding a function in $L^2(0, 1)$ and, then, they use Muntz polynomials to construct an approximation for f.

Now, in the present paper, we shall convert (DIL) to an analytic interpolation problem on the Hardy space of the unit disc. After that, we shall use Laguerre polynomials and coefficients of Lagrange polynomials to construct the function f. An approximation corresponding to the non exact data and error estimate will be given.

The remainder of the paper divided into two sections. In Section 2, we convert our problem into an interpolation one and give a uniqueness result. In Section 3, we shall give two regularization results in the cases of exact data and non exact data. Numerical comparisons with exact solution are given in the last section.

2. A uniqueness result

In this paper we shall use Laguerre polynomials

$$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^n).$$

We note that $\{L_n\}$ is a sequence of orthonormal polynomials on $L_\rho^2(0, \infty)$. We note that (see [1], [9], page 67)

$$\exp\left(\frac{xz}{z-1}\right) (1-z)^{-1} = \sum_{n=0}^{\infty} L_n(x) z^n.$$

Hence, if we have the expansion

$$f(x) = \sum_{n-0}^{\infty} a_n L_n(x)$$

then

$$\int_0^{\infty} f(x) \exp\left(\frac{xz}{z-1}\right) (1-z)^{-1} e^{-x} dx = \sum_{n=0}^{\infty} a_n z^n.$$

It follows that

$$\sum_{n=0}^{\infty} a_n z^n = \int_0^{\infty} f(x) \exp\left(\frac{x}{z-1}\right) (1-z)^{-1} dx.$$

Put $\Phi f(z) = \sum_{n=0}^{\infty} a_n z^n$, $\alpha_j = 1 - 1/p_j$, one has

$$\Phi f(\alpha_j) = p_j \mu_j,$$

i.e., we have an interpolation problem of finding an analytic function Φf in the Hardy space $H^2(U)$. Here, we denote by U the unit disc of the complex plane and by $H^2(U)$ the Hardy space. In fact, we recall that $H^2(U)$ is the space of all functions ϕ analytic in U and if, $\phi \in H^2(U)$ has the expansion $\phi(z) = \sum_{k=0}^{\infty} a_k z^k$ then

$$\|\phi\|_{H^2(U)}^2 = \sum_{k=0}^{\infty} |a_k|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\phi(e^{i\theta})|^2 d\theta.$$

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We can verify directly that the linear operator Φ is an isometry from L^2_ρ onto $H^2(U)$. In fact, we have

Lemma 1 *Let $f \in L^2_\rho(0, \infty)$. Then $\mathcal{L}f(z)$ is analytic on $\{z \in \mathbf{C} | \ Rez > 1/2\}$. If we have an expansion*

$$f = \sum_{n=0}^{\infty} a_n L_n$$

then one has $\Phi f \in H^2(U)$ and

$$\|\Phi f\|^2_{H^2(U)} = \sum_{n=0}^{\infty} |a_n|^2 = \|f\|^2_{L^2_\rho(0,\infty)}.$$

Moreover, If we have in addition that $\sqrt{x}f' \in L^2_\rho$ then

$$\sum_{n=0}^{\infty} n|a_n|^2 \leq \|\sqrt{x}f'\|^2_{L^2_\rho}.$$

Proof

Putting $F_z(t) = e^{-zt}f(t)$, we have $F_z \in L^2(0,\infty)$ for every $Rez > 1/2$. Hence $\mathcal{L}f(z) = \int_0^\infty F_z(t)dt$ is analytic for $Rez > 1/2$. From the definitions of $L^2_\rho(0,\infty)$ and $H^2(U)$, we have the isometry equality. Now we prove the second inequalities. We first consider the case f', f'' in the space

$$B = \{g \ Lebesgue \ measurable \ on \ (0,\infty) | \ \sqrt{x}g \in L^2_\rho(0,\infty)\}.$$

We have the expansion

$$f = \sum_{n=0}^{\infty} a_n L_n$$

where $a_n = \ <f, L_n>$.

The function $y = L_n$ satisfies the following equation (see [17])

$$xy'' + (1-x)y' + ny = 0$$

which gives

$$(xe^{-x}y')' + nye^{-x} = 0.$$

It follows that

$$
\begin{aligned}
na_n &= \int_0^\infty f(x)nL_n(x)e^{-x}dx \\
&= -\int_0^\infty f(x)(xe^{-x}L'_n(x))'dx \\
&= \int_0^\infty f'(x)xe^{-x}L'_n(x)dx \\
&= -\int_0^\infty (f'(x)xe^{-x})'L_n(x)dx \\
&= -\int_0^\infty (xf''(x) + f'(x) - xf'(x))L_n(x)e^{-x}dx \\
&= -<xf'' + f' - xf', L_n>.
\end{aligned}
$$

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Since L_n is an orthonormal basis, we have the Fourier expansion

$$xf" + f' - xf' = \sum_{n=0}^{\infty}(-na_n)L_n.$$

Using the Parseval equality we have

$$< xf" + f' - xf', f > = \sum_{n=0}^{\infty}(-na_n)a_n.$$

It can be rewritten as

$$\int_0^{\infty}(xe^{-x}f'(x))'f(x)dx = -\sum_{n=0}^{\infty}na_n^2.$$

Integrating by parts, we get

$$\int_0^{\infty}xe^{-x}|f'(x)|^2dx = \sum_{n=0}^{\infty}na_n^2.$$

Now, for $f' \in B$ we choose (f_k) such that $f'_k, f"_k \in B$ for every $k = 1, 2, ...$ and $\sqrt{x}f'_k$ (resp.f_k) $\to$ $\sqrt{x}f'$ (resp.f) in L_ρ^2 as $k \to \infty$. Assume that

$$f_k = \sum_{n=0}^{\infty}a_{kn}L_n.$$

Then we have

$$\int_0^{\infty}xe^{-x}|f'_k(x)|^2dx = \sum_{n=0}^{\infty}na_{kn}^2.$$

The latter equality involves for every N

$$\sum_{n=0}^{N}na_{kn}^2 \leq \|\sqrt{x}f'_k\|_{L_\rho^2(0,\infty)}^2 \tag{2}$$

Since $f_k \to f$ in L_ρ^2 as $k \to \infty$ we have that $a_{kn} \to a_n$ as $k \to \infty$, for each n. On the other hand, we have $\sqrt{x}f'_k \to \sqrt{x}f'$ in L_ρ^2 as $k \to \infty$. . Therefore, letting $k \to \infty$ in (2) we get

$$\sum_{n=0}^{N}na_n^2 \leq \|\sqrt{x}f'\|_{L_\rho^2(0,\infty)}^2.$$

Letting $N \to \infty$ in the latter inequality, we get the desired inequality. ∎

Using Lemma 1, one has a uniqueness result

Theorem 1. *Let $p_j > 1/2$ for every $j = 1, 2, ...$ If*

$$\sum_{p_j>1}\frac{1}{p_j} + \sum_{1/2<p_j<1}\frac{2p_j-1}{p_j} = \infty$$

then Problem (DIL) has at most one solution in $L^2_\rho(0, \infty)$.

Proof

Let $f_1, f_2 \in L^2_\rho(0, \infty)$ be two solutions of (DIL). Putting $g = f_1 - f_2$ then $g \in L^2_\rho(0, \infty)$ and $\mathcal{L}g(p_j) = 0$. It follows that $\Phi g(1 - 1/p_j) = 0$, $j = 1, 2, ...$ It follows that $\alpha_j = 1 - 1/p_j$ are zeros of Φg. We have $\Phi g \in H^2(U)$ and

$$\sum_{j=1}^{\infty}(1 - |\alpha_j|) = \sum_{p_j > 1} \frac{1}{p_j} + \sum_{1/2 < p_j < 1} \frac{2p_j - 1}{p_j} = \infty.$$

Hence we get $\Phi g \equiv 0$ (see, e.g.,[19], page 308). It follows that $g \equiv 0$. This completes the proof of Theorem 1. ∎

3. Regularization and error estimates

In the section, we assume that (p_j) is a bounded sequence, $p_j \neq p_k$ for every $j \neq k$. Without loss of generality, we shall assume that $\rho = 1$ is an accumulation point of p_j. In fact, if p_j has an accumulation point $\rho_0 > 1$ then, by putting $\tilde{f}(x) = e^{-(\rho_0 - 1)x}f(x)$ and $p'_j = p_j - \rho_0 + 1$, we can transform the problem to the one of finding $\tilde{f} \in L^2_\rho(0, \infty)$ such that

$$\int_0^\infty e^{-p'_j x}\tilde{f}(x)dx = \mu_j, \qquad j = 1, 2, ...$$

in which p'_j has the accumulation point $\rho = 1$. In fact, in Theorem 2 below, we shall assume that $\left|1 - \frac{1}{p_j}\right| \leq \sigma$ for every $j = 1, 2, ...$, where σ is a given number.

We denote by $\ell_k^{(m)}(\nu)$ the coefficient of z^k in the expansion of the Lagrange polynomial $L_m(\nu)$ ($\nu = (\nu_1, ..., \nu_m)$) of degree (at most) $m - 1$ satisfying

$$L_m(\nu)(z_k) = \nu_k, \qquad 1 \leq k \leq m,$$

where $z_k = \alpha_k$. If ϕ is an analytic function on U, we also denote

$$L_m(\phi) = L_m(\phi(z_1), ..., \phi(z_m)).$$

We define

$$L_m^\theta(\nu)(z) = \sum_{0 \leq k \leq \theta(m-1)} \ell_k^{(m)}(\nu)z^k.$$

The polynomial $L_m^\theta(\nu)$ is called a truncated Lagrange polynomial (see also [25]). For every $g \in L^2_\rho(0, \infty)$, we put

$$
\begin{aligned}
T_n g &= (p_1 \mathcal{L}g(p_1), ..., p_n \mathcal{L}g(p_n)), \\
Tg &= (p_1 \mathcal{L}g(p_1), ..., p_n \mathcal{L}g(p_n), ...) \in \ell^\infty.
\end{aligned}
$$

Here, we recall that $\alpha_n = 1 - 1/p_n$. We shall approximate the function f by

$$F_m = \Phi^{-1}L_m^\theta(T_m f) = \sum_{0 \leq k \leq \theta(m-1)} \ell_k^{(m)}(T_m f)L_k.$$

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We shall prove that F_m is an approximation of f. Before stating and proving the main results, some remarks are in order.

We first recall the concept of regularization. Let f be an exact solution of (DIL), we recall that a sequence of linear operator $A_n : \ell^\infty \to L^2_\rho(0,\infty)$ is a regularization sequence (or a regularizer) of Problem (DIL) if (A_n) satisfies two following conditions (see, e.g., [14], page 25)

(R1) For each n, A_n is bounded,

(R2) $\lim_{n\to\infty} \|A_n(Tf) - f\| = 0$.

The number "n" is called the *regularization parameter*. As a consequence of (R1), (R2), we can get

(R3) For $\epsilon > 0$, there exists the functions $n(\epsilon)$ and $\delta(\epsilon)$ such that $\lim_{\epsilon\to 0} n(\epsilon) = \infty$, $\lim_{\epsilon\to 0} \delta(\epsilon) = 0$ and that

$$\|A_{n(\epsilon)}(\mu) - f\| \le \delta(\epsilon)$$

for every $\mu \in \ell^\infty$ such that $\|\mu - Tf\|_\infty < \epsilon$.

In the present paper, the operator A_n is $\Phi^{-1} L^\theta_m$. The number ϵ is the error between the exact data Tf and the measured data μ. For a given error ϵ, there are infinitely many ways of choosing the regularization parameter $n(\epsilon)$. In the present paper, we give an explicit form of $n(\epsilon)$.

Next, in our paper, we have the interpolation problem of reconstruction the analytic function $\phi = \Phi f \in H^2(U)$ from a sequence of its values $(\phi(\alpha_n))$. As known, the convergence of $L_m(\phi)$ to ϕ depends heavily on the properties of the points (α_n). The Kalmár-Walsh theorem (see, e.g.,[12], page 65) shows that $L_m(\phi) \to \phi$ for every ϕ in $C(\overline{U})$ for all ϕ analytic in a *neighborhood* of $\overline{U}$ if and only if (α_n) is uniformly distributed in $\overline{U}$, i.e.,

$$\lim_{m\to\infty} \sqrt[n]{\max_{|z|\le 1} |(z - \alpha_1)...(z - \alpha_m)|} = 1.$$

The Fejer points and the Fekete points are the sequences of points satisfying the latter condition (see [12], page 67). The Kalmár-Walsh fails if $C(\overline{U})$ is replaced by $H^2(U)$ (see [25] for a counterexample). Hence, the Lagrange polynomial cannot use to reconstruct ϕ. In [12], we proved a theorem similar to the Kalmár-Walsh theorem for the case of $H^2(U)$. In fact, the Lagrange polynomials will convergence if we "cut off" some terms of the Lagrange polynomial. Especially, in [12] and the present paper, the points (α_n) are, in general, not uniformly distributed.

In Theorem 2, we shall verify the condition (R2). More precisely, we have

Theorem 2 *Let $\sigma \in (0, 1/3)$, let $f \in L^2_\rho(0, \infty)$ and let $p_j > 1/2$ for $j = 1, 2, ...$ satisfy*

$$\left| 1 - \frac{1}{p_j} \right| \le \sigma.$$

Put θ_0 be the unique solution of the equation (unknown x)

$$\frac{2\sigma^{1-x}}{1 - \sigma} = 1.$$

Then for $\theta \in (0, \theta_0)$, one has

$$\|f - F_m\|^2_{L^2_\rho} \longrightarrow 0 \quad \text{as } m \to \infty.$$

If, we assume in addition that $\sqrt{x} f' \in L^2_\rho(0, \infty)$ then

$$\|f - F_m\|^2_{L^2_\rho} \le (1 + m\theta)^2 \|f\|^2_{L^2_\rho} \left(\frac{2\sigma^{1-\theta}}{1 - \sigma} \right)^{2m} + \frac{1}{m\theta} \|\sqrt{x} f'\|^2_{L^2_\rho(0,\infty)}.$$

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Proof

We have in view of Lemma 1

$$\|f - F_m\|_{L_\rho^2}^2 = \sum_{0 \le k \le \theta(m-1)} |\delta_k^{(m)}|^2 + \sum_{k > \theta(m-1)} |a_k|^2 \tag{3}$$

where $\delta_k^{(m)} = a_k - \ell_k^{(m)}(T_m f)$. We shall give an estimate for $\delta_k^{(m)}$. In fact, we have

$$\|\Phi f - L_m(T_m f)\|_{H^2(U)}^2 = \sum_{k=0}^{m-1} |\delta_k^{(m)}|^2 + \sum_{k=m}^{\infty} |a_k|^2.$$

On the other hand, the Hermite representation (see, e.g. [12], page 59, [24]) gives

$$\Phi f(z) - L_m(T_m f)(z) = \frac{1}{2\pi i} \int_{\partial U} \frac{\omega_m(z)(\Phi f)(\zeta) d\zeta}{\omega_m(\zeta)(\zeta - z)}$$

where $\omega_m(z) = (z - \alpha_1)...(z - \alpha_m)$. Now, if we denote by $\sigma_{-1}^{(m)} = \sigma_{-2}^{(m)} = ... = 0$ and

$$\begin{aligned}
\sigma_0^{(m)} &= 1 \\
\sigma_r^{(m)} &= \sum_{1 \le j_1 < ... < j_r \le m} \alpha_{j_1}...\alpha_{j_r} \quad (1 \le r \le m), \\
\beta_s^{(m)} &= \frac{1}{2\pi i} \int_{\partial U} \frac{\Phi f(\zeta) d\zeta}{\zeta^{s+1} \omega_m(\zeta)}
\end{aligned}$$

then we can write in view of the Hermite representation

$$\Phi f(z) - L_m(T_m f)(z) = \sum_{k=0}^{\infty} \left(\sum_{r=0}^{k} (-1)^r \sigma_{m-r}^{(m)} \beta_{k-r}^{(m)} \right) z^k.$$

From the latter representation, one gets

$$\delta_k^{(m)} = \sum_{r=0}^{k} (-1)^r \sigma_{m-r}^{(m)} \beta_{k-r}^{(m)}, \quad 0 \le k \le m-1.$$

Now, by direct computation, one has

$$|\beta_s^{(m)}| \le \frac{1}{2\pi} \int_0^{2\pi} \frac{|\Phi f(e^{i\theta})|}{|\omega_m(e^{i\theta})|} d\theta.$$

But one has

$$|\omega_m(e^{i\theta})| \ge (|e^{i\theta}| - |\alpha_1|)...(|e^{i\theta}| - |\alpha_m|) \ge (1 - \sigma)^m.$$

Hence

$$\begin{aligned}
|\beta_s^{(m)}| &\le \frac{1}{2\pi(1-\sigma)^m} \int_0^{2\pi} |\Phi f(e^{i\theta})| d\theta \\
&\le \|\Phi f\|_{H^2(U)} (1 - \sigma)^{-m}.
\end{aligned}$$

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We also have
$$|\sigma_{m-r}^{(m)}| \leq \sigma^{m-r} C_m^r \leq \sigma^{m-k} 2^m,$$
where $C_m^k = \frac{m!}{k!(m-k)!}$. Hence, we have

$$|\delta_k^{(m)}| \leq (1 + m\theta)\|f\|_{L_\rho^2} \left(\frac{2\sigma^{1-\theta}}{1-\sigma}\right)^m.$$

From the latter inequality, one has in view of (3)

$$\|f - F_m\|_{L_\rho^2}^2 \leq (1 + m\theta)^2 \|f\|_{L_\rho^2}^2 \left(\frac{2\sigma^{1-\theta}}{1-\sigma}\right)^{2m} + \sum_{k \geq m\theta}^{\infty} |a_k|^2.$$

For $\theta \in (0, \theta_0)$, one has
$$0 < \frac{2\sigma^{1-\theta}}{1-\sigma} < \frac{2\sigma^{1-\theta_0}}{1-\sigma} = 1.$$

Hence, we have
$$\lim_{m \to \infty} \|f - F_m\|_{L_\rho^2}^2 = 0$$

as desired, since on the one hand we have the comparison between an exponential with base $b < 1$ and a power function and in the other hand the remain of a convergent series $\sum_{k=0}^{\infty} |a_k|^2$.

Now if $\sqrt{x} f' \in L_\rho^2(0, \infty)$ then one has since $\frac{k}{m\theta} > 1$ and from Lemma 1

$$\sum_{k>m\theta}^{\infty} |a_k|^2 \leq \frac{1}{m\theta} \sum_{k=0}^{\infty} k |a_k|^2 \leq \frac{1}{m\theta} \|\sqrt{x} f'\|_{L_\rho^2}^2.$$

This completes the proof of Theorem 2. ∎

Now, we consider the case of non-exact data. In Theorem 3, we shall consider the condition (R3) of the definition of the regularization. Put

$$D_m = \max_{1 \leq n \leq m} \left(\max_{|z| \leq R} \left| \frac{\omega_m(z)}{(z - \alpha_n)\omega_m'(\alpha_n)} \right| \right).$$

Let $\psi : [0, \infty) \to \mathbf{R}$ be an increasing function satisfying

$$\psi(m) \geq m D_m, \quad m = 1, 2, \dots$$

and

$$m(\epsilon) = [\psi^{-1}(\epsilon^{-3/4})] - 1$$

where $[x]$ is the greatest integer $\leq x$.

Theorem 3. *Let $\sigma \in (0, 1/3)$, let $f, \sqrt{x} f' \in L_\rho^2(0, \infty)$ and let $p_j > 1/2$ for $j = 1, 2, \dots$ satisfy*

$$\left| 1 - \frac{1}{p_j} \right| \leq \sigma.$$

Put θ_0 be the unique solution of the equation (unknown x)

$$\frac{2\sigma^{1-x}}{1-\sigma} = 1.$$

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Let $\epsilon > 0$ and let (μ_j^ϵ) be a measured data of $(\mathcal{L}f(p_j))$ satisfying

$$\sup_j |p_j(\mathcal{L}f(p_j) - \mu_j^\epsilon)| < \epsilon.$$

Then for $\theta \in (0, \theta_0)$, one has

$$\|f - \Phi^{-1}L_{m(\epsilon)}^\theta(\nu^\epsilon)\|_{L_\rho^2}^2 \le 2(1 + m(\epsilon)\theta)^2\|f\|_{L_\rho^2}^2 \left(\frac{2\sigma^{1-\theta}}{1-\sigma}\right)^{2m(\epsilon)} + \frac{2}{m(\epsilon)\theta}\|\sqrt{x}f'\|_{L_\rho^2}^2 + 2\epsilon^{1/2}.$$

where $\nu_j^\epsilon = p_j\mu_j^\epsilon$ for $j = 1, 2, ...$

Proof

We note that

$$L_m(T_mf)(z) - L_m(\nu^\epsilon)(z) = \sum_{j=1}^m (p_j\mu_j - \nu_j^\epsilon)\frac{\omega_m(z)}{(z-\alpha_j)\omega_m'(\alpha_j)}.$$

It follows that

$$\|L_m(T_mf) - L_m(\nu^\epsilon)\|_\infty \le \epsilon m D_m.$$

Hence

$$\|L_m^\theta(T_mf) - L_m^\theta(\nu^\epsilon)\|_{H^2(U)} \le \|L_m(T_mf) - L_m(\nu^\epsilon)\|_\infty \le \epsilon m D_m.$$

It follows by the isometry property of Φ

$$\begin{aligned}
\|f - \Phi^{-1}L_m^\theta(\nu^\epsilon)\|_{L_\rho^2}^2 &\le 2\|f - F_m\|_{L_\rho^2}^2 + 2\|\Phi^{-1}L_m^\theta(T_mf) - \Phi^{-1}L_m^\theta(\nu^\epsilon)\|_{L_\rho^2}^2 \\
&\le 2(1 + m\theta)^2\|f\|_{L_\rho^2}^2 \left(\frac{2\sigma^{1-\theta}}{1-\sigma}\right)^{2m} + \frac{2}{m\theta}\|\sqrt{x}f'\|_{L_\rho^2}^2 \\
&\quad + 2\epsilon^2 m^2 D_m^2.
\end{aligned}$$

By choosing $m = m(\epsilon)$ we get the desired result. ∎

4. Numerical results

We present some results of numerical comparison between the function $f(x)$ given in $L_\rho^2(0, \infty)$ and its approximated form F_m as it is stated in Theorem 2.

First consider the function $f(x) = e^{-x}$ and its expansion in Laguerre series

$$e^{-x} = \sum_{n \ge 0} \frac{1}{2^{n+1}}L_n(x). \tag{4}$$

So in the Hardy space $H^2(U)$, we have to interpolate the analytic function

$$\Phi f(x) = \sum_{n \ge 0} \frac{1}{2^{n+1}}x^n = \frac{1}{2 - x} \tag{5}$$

by the Lagrange polynomial $L_m(T_mf)$, interpolation defined by

$$L_m(T_mf)\left(1 - \frac{1}{p_i}\right) = p_i\int_0^\infty e^{-p_ix}e^{-x}dx = \frac{p_i}{p_i + 1} \tag{6}$$

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where $p_i \longrightarrow 1$ as $i \to \infty$.

On the interval $(-1.8, +1.8)$ we have drawn in Fig.1 the curves e^{-x} and its approximation $L_m(T_m f)(x)$ for $m = 10$. If $m = 12$ there is divergence for our interpolation (Fig.2) outside the interval $(-1, +1)$.
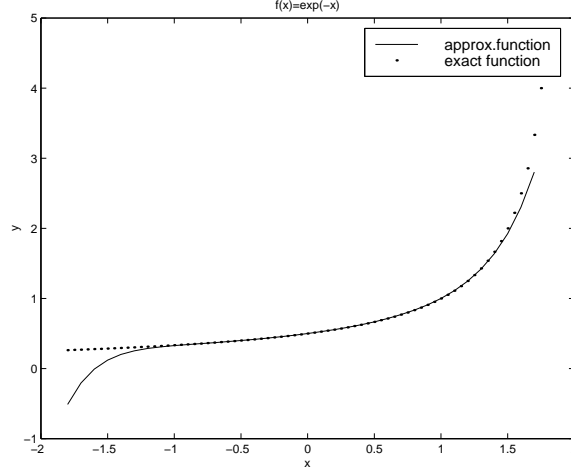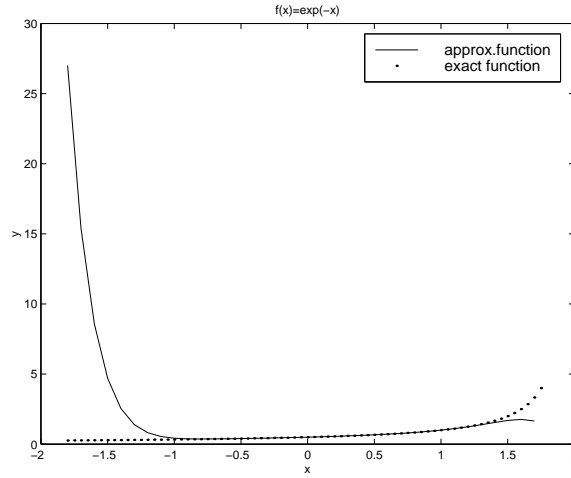

Fig. 1

Fig. 2

In our 2nd example we have chosen the function

$$f(x) = e^{x/4} = \frac{4}{3} \sum_{n \geq 0} \left(\frac{-1}{3} \right)^n L_n(x). \tag{7}$$

In the Hardy space the function

$$\Phi f(x) = \frac{4}{3} \sum_{n \geq 0} \left(\frac{-x}{3} \right)^n = \frac{4}{3 + x}$$

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is approximated by the Lagrange polynomial $L_m(T_m f)$ at the points $(1 - \frac{1}{p_i}, \frac{-4p_i}{1 - 4p_i})$, $p_i \longrightarrow 1$ as $i \to \infty$.

The Fig.3 (resp.Fig.4) show the quite good convergence (resp.divergence) on the interval $(-2.8, 2.8)$ with $m = 4$ (resp. $m = 11$).
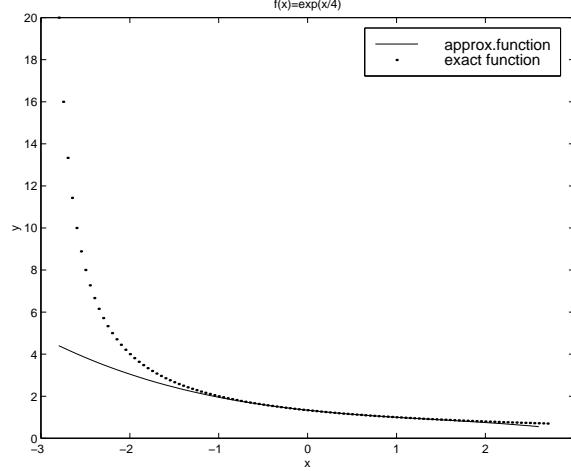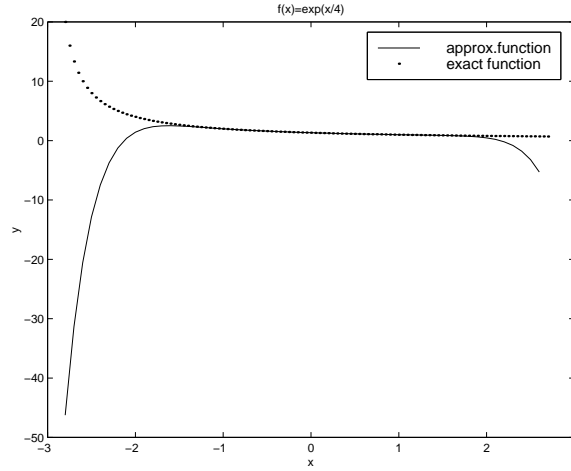

Fig.3

Fig. 4

In both cases we have chosen $\theta_0 = 0.29$ with $\sigma = 0.25$ (θ_0 given by $\frac{2\sigma^{1-\theta_o}}{1 - \sigma} = 1$, $0 < \sigma < \frac{1}{3}$). So in the 2nd case the truncated Lagrange polynomial is almost verified since $11 \times 0.29 \sim 3.2$.

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